February 2, 1865.

Major-General SABINE, President, in the Chair.

The following communications were read:—

I. "On a New Geometry of Space." By JULIUS PLÜCKER, For. Memb. R.S. Received December 22, 1864.

(Abstract.)

Infinite space may be considered either as consisting of points or transversed by planes. The points, in the first conception, are determined by their coordinates, by x, y, z for instance, taken in the ordinary signification; the planes, in the second conception, are determined in an analogous way by their coordinates, introduced by myself into analytical geometry, by t, u, v for instance. The equation

$$tx+uy+vz=1$$

represents, in regarding x, y, z as variable, t, u, v as constant, a plane by means of its points. The three constants t, u, v are the coordinates of this plane. The same equation, in regarding t, u, v as variable, x, y, z as constant, represents a point by means of planes passing through it. The three constants x, y, z are the coordinates of this point.

The geometrical constitution of space, referred hitherto either to points or to planes, may as well be referred to *right lines*. According to the double definition of such lines, there occurs to us a double construction of space. In the first construction we imagine infinite space to be traversed by lines, themselves consisting of points; an infinite number of such lines in all directions pass through any given point; the point may describe each of the lines. This constitution of space is admitted when, in optics, we regard luminous points sending out in all directions rays of light, or, in mechanics, forces acting on points in any direction. In the second construction, infinite space is regarded likewise as traversed by right lines, but these lines are determined by planes passing through them. Every plane contains an infinite number of lines having within it every position and direction, round each of which the plane may turn. We refer to this second construction when, in optics, we regard, instead of rays, the corresponding fronts of waves and their consecutive intersections, or when, in mechanics, according to Poinsot's ingenious philosophical views, we introduce into its fundamental principles "couples," as well entitled to occupy their place as ordinary forces. The instantaneous axes of rotation are right lines of the second description.

The position of a right line depends upon four constants, which may be determined in a different way. I adopted for this purpose the ordinary system of three axes of coordinates. A line of the first description, which

we shall distinguish by the name of *ray*, may be determined by means of two projections, for instance by those within XZ and YZ, represented by

$$x = rz + \rho,$$
$$y = sz + \sigma,$$

or by

$$tx + v_x z = 1,$$
$$uy + v_y z = 1.$$

In admitting the first system of equations, a ray is determined in a linear way by means of the four constants r, s, ρ, σ, which may be called its four coordinates, two of them, r and s, indicating its direction, the remaining two, after its direction being determined, its position in space. In adopting the second pair of equations, t, u, v_x, v_y will be the coordinates of the ray.

A right line of the second description, which we shall distinguish by the name of *axis*, is determined by any two of its points. It is the common intersection of all planes passing through both points. We may select the intersection of the axis with the two planes, XZ and YZ, as two such points, and represent them by

$$xt + z_t v = 1,$$
$$yu + z_u v = 1,$$

or by

$$t = pv + \pi,$$
$$u = qv + \kappa.$$

In making use of the first pair of equations, the four constants x, y, z_t, z_u, indicating the position of the two points within XZ and YZ, are the coordinates of the axis. In adopting the second pair, the four coordinates of the axis are p, q, π, κ.

A complex of rays or axes is represented by means of a single equation between their four coordinates; a *congruency*, containing all congruent lines of two complexes, by means of two such equations; a *configuration*, containing the right lines common to three complexes, by three equations. In a complex every point is the vertex of a cone, every plane contains an enveloped curve. In a congruency there is a certain number of right lines passing as well through a given point as confined within a given plane. A configuration is generated by a moving right line.

In a *linear complex* the right lines passing through a given point constitute a plane; all right lines within a given plane pass through a fixed point. Two linear complexes intersect each other along a *linear congruency*. In such a linear congruency there is a single right line passing as well through a given point as confined within a given plane. Three linear complexes meet along a *linear configuration*.

Instances of linear complexes are obtained by means of linear equations between the four coordinates of any one of the four systems. A linear configuration of rays represented by three such equations between r, s, ρ, σ is a paraboloid, immediately obtained; between t, u, v_x, v_y a hyperboloid.

A linear configuration of axes represented by three linear equations between p, q, π, κ is a hyperboloid, immediately obtained; between x, y, z_t, z_u a paraboloid. Instances of linear congruencies are exhibited by means of two linear equations, as well between t, u, v_x, v_y as between x, y, z_t, z_u, and their right lines easily constructed.

The general linear equation, however, between any four coordinates does not represent a linear complex of the most general description. Besides, there is a want of symmetry, the four coordinates depending upon the choice of both planes, XZ and YZ. This double inconvenience, if not eliminated, would render it impossible to adapt in a proper way analysis to the new geometrical conception of space. But it may be eliminated in the most satisfactory way.

For that purpose I introduced (in confining myself to the case of the coordinates r, s, ρ, σ) a fifth coordinate $(s\rho-r\sigma)$, which is a function of the four primitive ones. Then the linear equation between the five coordinates

$$\mathrm{A}r+\mathrm{B}s+\mathrm{C}+\mathrm{D}\sigma+\mathrm{E}\rho+\mathrm{F}(s\rho-r\sigma)=0$$

is the most general of a linear complex. After having been rendered homogeneous by a sixth variable introduced, it becomes of a complete symmetry with regard to the three axes OX, OY, OZ. The introduction of the fifth coordinate $(s\rho-r\sigma)$ is the real basis of the new analytical geometry, the exploration of which is indicated in the ordinary way.

In the paper presented, a complete analytical discussion of a linear complex is given. We may for any point of space construct the *corresponding plane* containing all traversing rays, and *vice versâ*. Right lines of space associate themselves into couples of *conjugated lines*; to each line a conjugated one corresponds. Any right line intersecting any two conjugated, is a ray of the complex. Each ray of it is to be regarded as two coincident conjugated lines. It is easily shown that each linear complex may be represented by means of any one of the following three equations, in which k indicates the same constant:

$$s\rho-r\sigma=k,\quad \sigma=kr,\quad \rho=ks.$$

Accordingly a linear complex depends upon the position of a fixed line (depending itself upon four constants) and the constant k. Hence it likewise follows that such a complex of rays may, without being changed, as well turn round that fixed line, *the axis of the complex*, as move along it, parallel to itself. The same results may be confirmed by means of the transformation of ray-coordinates, and thus analytically determined by the primitive constants A, B, C, D, E, F, the position of the axis of the complex and its constant k. In a peculiar case, where k becomes *zero*, all rays of the complex meet its axis.

A linear congruency of rays, along which an infinite number of linear complexes meet, is represented by the equations of any two of these complexes. Through a given point of space passes only one ray, *corresponding* to it, as there is only one corresponding ray confined within a given

plane. There is, with regard to each complex passing through the congruency, one right line conjugated to a given one. All these conjugated lines constitute one generation of a hyperboloid, while the right lines of its other generation are rays of the congruency, which therefore may be generated by a variable hyperboloid turning round one of its right lines.

The axes of all complexes intersecting each other along a linear congruency meet at right angles a fixed line, which is the axis of the congruency. Among the complexes there are especially two, the axes of which are met by their rays. These axes, meeting themselves the axis of the congruency, are its directrices. A linear congruency, depending upon eight constants, is fully determined by means of its two *directrices*. Each right line intersecting both directrices is one of its rays. The plane parallel to both directrices, and at equal distance from them, is the *central plane* of the congruency; the point where it meets, under right angles, the axis of the congruency, its *centre*. The two lines bisecting within the central plane the projections of the two directrices, are its *secondary axes*. The directrices may be as well both *real* as both *imaginary*. In peculiar cases the two directrices are congruent, or one of them is at an infinite distance. Each of two complexes being given by means of its axis and its constant k, both directrices of the congruency along which they intersect one another are analytically determined. A congruency being given by means of its directrices, the constants and axes of all complexes passing through it are determined.

A *linear configuration of rays* is the common intersection of any three linear complexes, and represented by their equations, $\Omega=0$, $\Omega'=0$, $\Omega''=0$. Each complex represented by an equation of the form $\Omega+\mu\Omega'+\nu\Omega'=0$, equally passes through the same configuration. So does any congruency along which two such complexes meet. A linear configuration is a hyperboloid; its rays constitute one of its generations, while the directrices of all traversing congruencies constitute the other. The central planes of all these congruencies meet in the same point—the *centre* of the hyperboloid. Its *diameters* meet both directrices of the different congruencies. The directrices are either real or imaginary; accordingly the diameters meet the hyperboloid, or meet it not. If the two directrices are congruent, the diameters become asymptotes. The hyperboloid passes into a paraboloid if there is one directrix infinitely distant.

A linear configuration is determined by means of three congruencies as it is by means of three complexes. That ray of it which meets one directrix of each congruency is parallel to the other. By drawing two planes through the two directrices of each of the three congruencies parallel to its central plane, we get a rhomboid circumscribed about the hyperboloid, the points of contact, within the six planes, being the points where the six directrices are intersected by the rays. A hyperboloid being given, we may revert to the congruencies and complexes constituting it. Finally, the equation of the hyperboloid in ordinary coordinates, x, y, z, is derived.

If we proceed to complexes of the second degree, the field of inquiry is immensely increased. Here any given point of infinite space is the vertex of a cone of the second order, and likewise within any given plane there is a curve of the second class enveloped by rays of the complex. The whole of the infinite number of cones, as well as of the infinite number of enveloped conics, is represented by a linear equation, between the five ray-coordinates r, s, ρ, σ and $(s\rho - r\sigma)$. The general analytical theory of contact may immediately be applied to complexes of the second order, touched by linear complexes, &c.

In order to elucidate the geometrical conceptions explained, I thought it proper to present, in Section II., an application to optics, leading to a complex of a simple description. Rays of light, constituting in air a complex, will likewise do so after being submitted to any reflexions or refractions whatever. Let us, for instance, suppose that the complex in air is of the first order and its constant equal to zero; *i. e.*, that its rays start in every direction from all points of a luminous right line. Let these rays enter a *biaxal crystal* by any plane surface. Let the luminous line and this surface be perpendicular to each other. Then, within the crystal, the double-refracted rays constitute a new complex, which is represented, like the primitive one, and independently of it, by means of an equation between ray-coordinates.

For this purpose I return to a paper of mine of the year 1838, concerning double refraction, at the end of which, after having mentioned the application of Huyghens's principle to Fresnel's wave-surface and the construction of Sir William Hamilton, I proposed a new construction of the double-refracted rays in the most general case. Here I first made use of an auxiliary ellipsoid, with regard to which the polar plane of every point of the wave-surface is one of its tangent planes, and, reciprocally, the pole of every plane touching the surface one of its points. In representing Fresnel's ellipsoid by the equation

$$a^2 x^2 + b^2 y^2 + c^2 z^2 = 1,$$

the new auxiliary ellipsoid may be represented by

$$\frac{x^2}{bc} + \frac{y^2}{ac} + \frac{z^2}{ab} = 1,$$

or

$$ax^2 + by^2 + cz^2 = abc,$$

and replaced, for most purposes, by the similar one,

$$ax^2 + by^2 + cz^2 = 1.$$

The construction, as far as we are concerned here, may be expressed thus:—Construct at the moment when Fresnel's wave-surface is formed the polar line of the trace along which the surface of the crystal is intersected by the elementary wave. The two refracted rays meet the wave-surface in the two points where it is intersected by the polar line constructed. In the paper of

1838 I promised a discussion of the construction given, but neglected it till the present time. This discussion immediately leads us to represent the complex of double-refracted rays by an equation, and at the same time we meet with several theorems worthy of notice.

If there is any incident ray, *the plane of refraction*, containing both double-refracted rays, is congruent with the diametral plane of the auxiliary ellipsoid, the conjugated diameter of which is perpendicular to the plane of incidence. All rays incident within the same plane are, after double refraction, confined again within the same plane. While the plane of incidence turns round the vertical, the corresponding plane of refraction turns round that diameter of the auxiliary ellipsoid, the conjugated diametral plane of which is the surface of the crystal. Whatever may be the plane or curved surface by which a crystal is bounded in a given point, all corresponding planes of refraction pass through a fixed right line.

A complex of rays starting in air in all directions from every point of a luminous right line, perpendicular to the surface of the crystal, is represented by the equation

$$r\sigma = s\rho,$$

the luminous right line being the axis OZ, while the two remaining axes, OX and OY, are within the surface of the crystal any two right lines perpendicular to each other. This complex is transformed by double refraction into another, the equation of which assumes the most simple form,

$$r\sigma = ks\rho,$$

in especially admitting that the two axes, OX and OY, are congruent with the axes of the ellipse along which the auxiliary ellipsoid is cut by the surface of the crystal, and that the third axis, OZ, is, within the crystal, the diameter of the auxiliary ellipsoid; the conjugated diametral plane is that surface. k is a constant indicating the ratio of the squares of the two axes of the ellipse.

The complex of double-refracted rays is of the second order; its equation may be easily submitted to analytical discussion. All its rays passing through any given point constitute a cone of the second order. This cone remains the same if the point describes a right line, passing through the origin. Likewise there is in any given plane a hyperbola, enveloped by rays of the complex. Peculiar cases are easily determined. The complex of double-refracted rays may be described in three different ways by a variable linear congruency. In the peculiar case in which the surface of the crystal is a principal section, OZ becomes perpendicular to it; if it is one of the circular sections of the auxiliary ellipsoid, the constant k becomes equal to unity, *i. e.* all double-refracted rays meet the axis OZ. From the general case the case of uniaxal crystals is immediately derived.